ON A VARIATIONAL THEOREM OF THE NONLINEAR THEORY OF ELASTICITY

(OB ODNOI VARIATSIONNOI TEOREME NELINEINOI TEORII UPRUGOSTI)

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Consider the functional

$$J = \iiint_{V_*} \mathbf{Q}^* \cdot \mathbf{u} dV_* + \iint_{S_*} \mathbf{P}^* \cdot \mathbf{u} dS_* - \iiint_{V_*} \left\{ W_* + F_* + \frac{1}{2} \sigma_*^{ik} \partial_i \mathbf{u} \cdot \partial_k \mathbf{u} \right\} dV_* \qquad (1)$$

where S_* is the boundary of a three dimensional domain V_*, occupied by a deformed body. Let V_* be parametrized by coordinates x^i ($i = 1, 2, 3$) with metric tensor g_{ik}^*. $\mathbf{Q}^*$ is the body force vector and $\mathbf{P}^*$ is the surface traction, referred to unit volume in V_* and to unit surface area in S_*, respectively; $\mathbf{u}$ is the displacement vector; $\sigma_*^{ik} = \sigma_*^{ki}$ are the contravariant components of the stress tensor, per unit area in the deformed body; $F_* = F_*(\sigma_*^{ik})$ is a function of the stress tensor; and $W = W_*(\epsilon_{ik}^*)$ is a function of the strain tensor $\epsilon_{ik}^* = \epsilon_{ki}^*$.

The variational theorem may be formulated as follows: among all the possible displacements $\mathbf{u}$, compatible with the geometrical constraints, among all stress components σ_*^{ik} which satisfy the statical conditions of equilibrium in the interior and on the boundary, and among all strain components ϵ_{ik}^*, the actual ones render stationary the functional J. Thus, the fulfillment of all the conditions of the nonlinear theory of elasticity implies that $\delta J = 0$, and, conversely, $\delta J = 0$ implies the fulfillment of all the relations of the nonlinear theory of elasticity.

The functional J may be rewritten as follows

$$J = \iiint_{V} \mathbf{Q} \cdot \mathbf{u} dV + \iint_{S} \mathbf{P} \cdot \mathbf{u} dS - \iiint_{V} \left\{ W + F + \frac{1}{2} \sigma^{ik} \partial_i \mathbf{u} \cdot \partial_k \mathbf{u} \right\} dV \qquad (2)$$

where

$$Q = Q_* \sqrt{\frac{g_*}{g}}, \quad P = P_* \sqrt{\frac{a_{(s)}{}^*}{a_{(s)}}}, \quad W = W_* \sqrt{\frac{g_*}{g}}, \quad F = F_* \sqrt{\frac{g_*}{g}},$$

$$\sigma^{ik} = \sigma_*{}^{ik} \sqrt{\frac{g_*}{g}}, \quad g_* = \det(g_{ik}{}^*), \quad g = \det(g_{ik}), \quad a_{(s)} = \det(a^{(s)}_{\alpha\beta}), \ldots \tag{3}$$

In Equation (2), S is the boundary of the volume V occupied by the body before deformation; $a^{(s)}_{\alpha\beta}$ is the metric tensor on the surface S; and g_{ik} is the metric tensor in the domain V with coordinates x^i, which is induced by the parametrization of the domain V_*.

The first variation of the functional J is

$$\delta J = \iiint_V Q \cdot \delta u \, dV + \iint_S P \cdot \delta u \, dS - \iiint_V \sigma^{ik} \delta e_{ik}{}^* \, dV + \iiint_V \left\{ \sigma^{ik} - \frac{\partial W}{\partial e_{ik}{}^*} \right\} \delta e_{ik}{}^* \, dV +$$
$$+ \iiint_V u \cdot \delta Q \, dV + \iint_S u \cdot \delta P \, dS - \iiint_V \left\{ \left(\frac{\partial F}{\partial \sigma^{ik}} + \frac{1}{2} \partial_i u \cdot \partial_k u \right) \delta\sigma^{ik} + \sigma^{ik} \partial_i u \cdot \delta r_k{}^* \right\} dV \tag{4}$$

where $r_i{}^*$, r_i are the coordinate vectors in the spaces V_* and V respectively $(r_i{}^* = r_i + \partial_i u \quad \delta r_i{}^* = \delta \partial_i u)$.

Since the variations of the stress do not violate the conditions of static equilibrium in the interior and on the boundary of the body, it follows that

$$\frac{\partial \delta (\sigma^{ik} r_k{}^* \sqrt{g})}{\sqrt{g}\, \partial x^i} + \delta Q = 0, \qquad \delta (\sigma^{ik} r_k{}^* n_i) + \delta P = 0 \tag{5}$$

Substituting these relations into (4) we have

$$\delta J = \iiint_V Q \cdot \delta u \, dV + \iint_S P \cdot \delta u \, dS - \iiint_V \sigma^{ik} \delta e_{ik}{}^* \, dV + \iiint_V \left\{ \sigma^{ik} - \frac{\partial W}{\partial e_{ik}{}^*} \right\} \delta e_{ik}{}^* dV -$$
$$- \iiint_V u \cdot \nabla_i \delta (\sigma^{ik} r_k{}^*) \, dV - \iint_S u \delta (\sigma^{ik} r_k{}^* n_i) dS -$$
$$- \iiint_V \left\{ \left(\frac{\partial F}{\partial \sigma^{ik}} + \frac{1}{2} \partial_i u \cdot \partial_k u \right) \delta\sigma^{ik} + \sigma^{ik} \partial_i u \cdot \delta r_k{}^* \right\} dV \tag{6}$$

where the n_i are the covariant components of the inner normal vector to S and $\nabla_i(\ldots)$ denotes the covariant derivative with respect to the metric g_{ik}.

By means of the following formula for integrating by parts

$$\iiint_V u \cdot \nabla_i \delta (\sigma^{ik} r_k{}^*) \, dV = \iiint_V \nabla_i \left[u \cdot \delta (\sigma^{ik} r_k{}^*) \right] dV - \iiint_V \delta (\sigma^{ik} r_k{}^*) \cdot \nabla_i u \, dV =$$
$$= - \iint_S u \cdot \delta (\sigma^{ik} r_k{}^*) \, n_i dS - \iiint_V \nabla_i u \cdot (r_k{}^* \delta\sigma^{ik} + \sigma^{ik} \delta r_k{}^*) \, dV \tag{7}$$

one may rewrite δJ as follows

$$\delta J = \iiint\limits_V \mathbf{Q}\cdot\delta\mathbf{u}\, dV + \iint\limits_S \mathbf{P}\cdot\delta\mathbf{u}\, dS - \iiint\limits_V \sigma^{ik}\delta\varepsilon_{ik}{}^* dV + \iiint\limits_V \left\{\sigma^{ik} - \frac{\partial W}{\partial\varepsilon_{ik}{}^*}\right\}\delta\varepsilon^*_{ik}\, dV -$$
$$- \iiint\limits_V \left\{\frac{\partial F}{\partial\sigma^{ik}} + \frac{1}{2}\,\partial_i\mathbf{u}\cdot\partial_k\mathbf{u} - \partial_i\mathbf{u}\cdot\mathbf{r}_k{}^*\right\}\delta\sigma^{ik} dV \tag{8}$$

Further, since

$$\partial_i\mathbf{u}\cdot\mathbf{r}_k{}^* = \partial_i\mathbf{u}\cdot(\mathbf{r}_k + \partial_k\mathbf{u}) = \nabla_i u_k + \nabla_i u_n \nabla_k u^n = \nabla_i{}^* u_k{}^* \tag{9}$$

one has

$$\delta J = \iiint\limits_V \mathbf{Q}\cdot\delta\mathbf{u}\, dV + \iint\limits_S \mathbf{P}\cdot\delta\mathbf{u}\, dS - \iiint\limits_V \sigma^{ik}\delta\varepsilon_{ik}{}^* dV + \iiint\limits_V \left\{\sigma^{ik} - \frac{\partial W}{\partial\varepsilon_{ik}{}^*}\right\}\delta\varepsilon_{ik}{}^* dV -$$
$$- \iiint\limits_V \left\{\frac{\partial F}{\partial\sigma^{ik}} - \frac{1}{2}\Big(\nabla_i u_k + \nabla_k u_i + \nabla_i u_n \nabla_k u^n\Big)\right\}\delta\sigma^{ik} dV \tag{10}$$

or

$$\delta J = \iiint\limits_V \mathbf{Q}\cdot\delta u\, dV + \iint\limits_S \mathbf{P}\delta\mathbf{u}\, dS - \iiint\limits_V \sigma^{ik}\delta\varepsilon_{ik}{}^* dV + \iiint\limits_V \left\{\sigma^{ik} - \frac{\partial W}{\partial\varepsilon_{ik}{}^*}\right\}\delta\varepsilon_{ik}{}^* dV -$$
$$- \iiint\limits_V \left\{\frac{\partial F}{\partial\sigma^{ik}} - \frac{1}{2}(\nabla_i{}^* u_k{}^* + \nabla_k{}^* u_i{}^* - \nabla_i{}^* u_n{}^* \nabla_k{}^* u_*{}^n)\right\}\delta\sigma^{ik} dV - \tag{11}$$

In this last equation $\mathbf{u} = u_i \mathbf{r}^i = u_i{}^* \mathbf{r}_*{}^i$, and $\nabla_i{}^*(\ldots)$ is the covariant derivative with respect to the metric $g_{ik}{}^*$.

Suppose that W is a potential for the stress tensor and that F is a potential for the strain tensor:

$$\sigma^{ik} = \frac{\partial W}{\partial\varepsilon_{ik}{}^*}, \qquad \varepsilon_{ik}{}^* = \frac{\partial F}{\partial\sigma^{ik}} \tag{12}$$

If the compatibility conditions for the displacements are satisfied, then the last integrals in (10) and (11) must vanish. The satisfaction of the elasticity relations carries along with it the vanishing of the last of the then remaining integrals in (10) and (11). The collection of the remaining integrals then gives nothing else but the variational equations of Lagrange, which are certainly satisfied once the equations of equilibrium and the boundary conditions are themselves fulfilled.

Thus, $\delta J = 0$ for the actual state. Conversely, from $\delta J = 0$ follow all the relations of the nonlinear theory of elasticity. Indeed, since $\delta\sigma^{ik}$ is independent of $\delta\epsilon_{ik}{}^*$ and of $\delta\mathbf{u}$, the equation $\delta J = 0$ gives, making use of the last integral of (11),

$$2\varepsilon_{ik}{}^* = \nabla_i{}^* u_k{}^* + \nabla_k{}^* u_i{}^* - \nabla_i{}^* u_n{}^* \nabla_k{}^* u^{*n} \tag{13}$$

or

$$2\varepsilon_{ik} = \nabla_i u_k + \nabla_k u_i + \nabla_i u_n \nabla_k u_n^* \tag{14}$$

From these equations it follows that [1]

$$\sigma^{ik}\, \delta\varepsilon_{ik}{}^* = \sigma^{ik} \mathbf{r}_i{}^* \frac{\partial \delta \mathbf{u}}{\partial x^k}.$$

Hence

$$\delta J = \iiint\limits_V \{\nabla_i(\sigma^{ik}\mathbf{r}_k{}^*) + \mathbf{Q}\}\, \delta\mathbf{u}\, dV + \iint\limits_S \{\sigma^{ik}\mathbf{r}_k{}^* n_i + \mathbf{P}\}\, \delta\mathbf{u}\, dS + $$
$$+ \iiint\limits_U \left\{\sigma^{ik} - \frac{\partial W}{\partial \varepsilon_{ik}{}^*}\right\} \delta\varepsilon_{ik}{}^*\, dV = 0 \tag{15}$$

and from this variational equation follow the equations of equilibrium

$$\triangle_i\, (\sigma^{ik}\, \mathbf{r}_k{}^*) + \mathbf{Q} = 0$$

and the natural static boundary conditions

$$\sigma^{ik}\, \mathbf{r}^{*k} n_i + \mathbf{P} = 0$$

while from the geometrical conditions $\delta\mathbf{u} = 0$ follow the corresponding equations of elasticity

$$\sigma^{ik} = \frac{\partial W}{\partial \varepsilon_{ik}{}^*}$$

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